’Particles’ and ’Waves’ as Understood by Temporal Concept Analysis

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Abstract. A mathematical framework for a granular representation of relational information about general objects, as for examples particles, waves, persons, families, institutions or other systems in discrete or continuous “space-time” is introduced. That leads to a common description of particles and waves in physics, such that the intuitive physical notion of “wave packet” can be captured by the mathematical notion of a *general object* or a *packet* in a *spatio-temporal Conceptual Ontology System*. In these systems *particles* and *waves* are defined as special packets. It is proved, that “classical objects” and “classical waves” can be represented as, respectively, *particles* and *waves* in spatio-temporal Conceptual Ontology Systems.
The underlying mathematical framework is Formal Concept Analysis and its subdiscipline Temporal Concept Analysis. That allows for describing continuously fine or arbitrarily coarse granularity in the same conceptual framework. The central idea for a “symmetric” description of objects, space, and time is the introduction of “instances” representing information units of observations, connecting the granules for objects, space, and time.

1 Granular Representation of Relational Information

This paper introduces a mathematical framework for an ontology based representation of relational information about general objects, including particles and waves in discrete or continuous “space-time”. In that framework ontologies are understood as well-accepted hierarchies of concepts, which will be represented here as concept lattices. That allows for describing continuously fine or arbitrarily coarse granularity in the same conceptual framework. Usual space-time theories are not concerned with a theoretical representation of granularity - the success of Euclidean geometry and the precise Newtonian laws led to the conviction that granularity is a good tool for the practitioner, but not for the theorist. Einstein was aware of that problem when he wrote his “granularity” footnote in [4], p. 893:

> The inaccuracy which lies in the concept of simultaneity of two events at (about) the same place and which has to be bridged also by an abstraction, shall not be discussed here.

In the following, we discuss the “inaccuracy which lies in the concept of simultaneity of two events” using a granularity notion for “time granules”; we also introduce a granularity notion for “space granules” and “object granules”. The mathematical background of Conceptual Knowledge Processing and Temporal Concept Analysis is described in the next section.

1.1 Conceptual Knowledge Processing and Temporal Concept Analysis

The mathematical core of Conceptual Knowledge Processing is *Formal Concept Analysis* as introduced by Wille [10, 11, 5]. We assume, that the reader is familiar with the basic notions in Formal Concept Analysis. For an introduction we refer to [11, 13].

In Temporal Concept Analysis (TCA) as developed by the author [14, 17, 18, 20, 21] "objects" are studied with respect to "space" and "time". For that purpose the author has introduced the notion of a *Conceptual Time System* which allows for defining *states*, *situations* [14], *transitions* [17], and *life tracks* [18, 21]. The central structure introduced in [18] is called a "Conceptual Time Systems with Actual Objects and a Time Relation" (CTSOT). The CTSOTs are designed for describing "timeless objects" in the sense, that the definition of a CTSOT starts with set P whose elements are called objects (or persons or particles). The connection of objects with time is then represented by taking the *actual objects* (p,g) where p is an object and g is a "time granule" as formal objects of a (many-valued) context. A binary *time relation* connecting the actual objects yields a first connection to the "genidentity" in the sense of Reichenbach [7] which can be formally described as a relation R between two actual objects (p,g) and (q,h) defined by (p,g) R (q,h) $\iff$ p = q and g < h, where "<" denotes a strict order relation on the set of time granules. That description of the "genidentity", which is not satisfactory, since it uses given "timeless objects", led the author to a conceptual investigation of "general objects" and their relation to "space" and "time" which allows for describing "usual objects" as well as "waves". That will be discussed in the next sections, but first we consider particles and waves in physics.

1.2 Particles in Physics

Particles in physics are usually understood as special objects - which does not clarify much, at least as long as we do not have a theoretical understanding of the notion of "object". There is a huge literature surrounding that theme. The interested reader is referred to the fine collection of important papers concerning the philosophical and quantum physical aspects of objects in [3, 2]. In her introduction Elena Castellani mentions basic philosophical questions around physical objects, the relation between part and whole, the problem of "elementary particles" as objects "without (proper) material parts", and the problem of describing the "parts of an object". Of special interest for our purpose is the problem of identity through time which ([3], p.7, Introduction by E. Castellani)

> can also be understood in the sense of searching for "something" that unites the successive parts or momentary stages of an object. In the literature, this "something" is usually called *genidentity*, after Hans Reichenbach's use of the term for characterizing the relation connecting different states of the same physical object at different times.
> In today's debate on physical objects, we can distinguish between two main theories of individuation, depending on whether the role of conferring individuality is ascribed to properties of the object, or to something "transcending" the object's set of properties, such as some kind of persisting substantial substratum or essence.

1.3 Waves in Physics

Waves in physics are usually described as solutions of the wave equation, in any case as functions; for example an electro-magnetic wave is described as a function assigning to each "location" (x,y,z) and to each "point of time" t a six-tuple of values (E_x,E_y,E_z,H_x,H_y,H_z) of the electro-magnetic field. If we just write down the 10 values as a 10-tuple

$$(x, y, z, t, E_x, E_y, E_z, H_x, H_y, H_z)$$

we forget to represent, that the last six values are functionally dependent on the first four values. Wherefrom do we know, that such a dependency holds for all kinds of waves? Clearly, it does not hold, since the amplitude z of a water wave which is breaking near the beach is not a function of the coordinates x,y,t - there may be more than only one surface point above (x,y) at time t. Why do we mention that phenomenon of the "breaking water wave"? First for emphasizing, that employing the notion of a function for describing waves theoretically should be seen as a pragmatical choice and not as the only possibility - another possibility is a relational approach. The second reason for mentioning the "breaking water wave" is, that the uniqueness of a value at a place and a time - in that example, the z-value of the wave surface over (x,y) at time t - clearly depends heavily on the granularity of the chosen description for "space", "time", and the domain for the "values". The third aspect of the "breaking water wave" is, that the measurement of "real phenomena" should be clearly distinguished from the description of these phenomena by "physical laws" (like the "wave equation"), which are well-suited for short representations of dependencies observed in granular data. In the following we focus on the measurement aspect with a theoretical treatment of granularity.

1.4 Particles, Waves, and Wave Packets

This subsection serves to describe some general similarities between particles and waves. These similarities are the key for understanding particles and waves in an ontology based granularity framework for general objects.

Abstracting from the usual physical understanding of particles to a general understanding of "objects" in "space" and "time" we now describe a property of "objects in space and time" which seems to be widely accepted, but which is not yet clear, since its granularity is not yet made explicit. We call it the

***Object Axiom*:**
Each object is at each time granule at exactly one place.

Using the CTSOTs [18] the object axiom can be proved easily with a granular notion for "space" and "time", but the CTSOTs do not provide a granularity for the "objects". The object axiom will play the leading role for defining objects in the following.

Similarly, we describe a widely accepted property of waves by the following

***Wave Axiom*:**
Each wave has at each place and each time granule exactly one value.

These two axioms seem to describe totally different entities, but we shall see, that they can be understood as two extremes of the same kind of entities, which will be called "wave packets" or for short "packets". For introducing them we first define Conceptual Ontology Systems in the next section.

2 Conceptual Ontology Systems

In this section we introduce Conceptual Ontology Systems (COS) with the purpose to use ontological granularity tools symmetrically for all "dimensions" as for example for “space”, “time” and also for “general objects”. That end can be attained by introducing a formal representation of “instances”, which are interpreted as “judgments”, expressing that something has been observed.

2.1 Ontologies, Observations, and Instances

For the purpose of representing well-accepted knowledge frames like for example the frames “plant - animal - living being” or “morning - day - evening - night”, or “place - town - country” we employ formal contexts, called “ontologies”. The formal concepts of these ontologies are used as basic elements for “instances” describing observations. If these instances are taken as formal objects of a many-valued context, whose values are the formal concepts of these ontologies, then we get a “Conceptual Ontology System” as defined in Definition 1.

Definition 1. ***“Conceptual Ontology System”***
Let M be a set, and for each $m \in M$ *let* $\mathbb{O}_m := (G_m, N_m, I_m)$ *be a formal context, called the ontology of m. Let* $(\mathfrak{B}_m, \leq_m)$ *be the concept lattice of the ontology* $\mathbb{O}_m$*, and* $\mathfrak{B} := \bigcup \{\mathfrak{B}_m \mid m \in M\}$*.*
Let (G, M, W, I) be a complete many-valued context such that

$$m(g) \in \mathfrak{B}_m \qquad (\text{for all } g \in G \text{ and } m \in M).$$

Then the pair$((\mathbb{O}_m \mid m \in M), (G, M, W, I))$ *is called a Conceptual Ontology System (COS). The elements of G are called instances.*

Remark: The ontologies $\mathbb{O}_m$ (m $\in$ M) are not conceptual scales of [4](G, M, W, I). The connection between Conceptual Ontology Systems and scaled many-valued contexts is quite close, as the following definition of the “ontology scaled context” and the “Ontology Scaling Lemma” demonstrates.

2.2 Ontology Scaling

Similarly to the construction of the derived context of a scaled many-valued context we now define the “ontology derived context” of a Conceptual Ontology System.

Definition 2. ***“ontology derived context”***
Let $((\mathbb{O}_m \mid m \in M), (G, M, W, I))$ *be a Conceptual Ontology System and let* $int(c)$ *denote the intent of a concept c. Then the formal context*

$$\mathbb{K}^o := (G, \{(m,n) \mid m \in M, n \in N_m\}, J^o) \text{ where}$$
$$gJ^o(m,n) :\Longleftrightarrow n \in int(m(g))$$

is called the ontology derived context of $((\mathbb{O}_m \mid m \in M), (G, M, W, I))$*.*

What is the connection to the procedure of plain scaling? Usually we interpret in a many-valued context (G, M, W, I) the formal objects $g \in G$ as “objects”, the many-valued attributes $m \in M$ as “measurements” and a triple $(g, m, w) \in I$ as the “instance”, that “the value of the measurement m at object g is w”. Then a scale $\mathbf{S}_m$ of an attribute $m \in M$ is a formal context, which is connected to (G, M, W, I) only by the week condition, that the set of formal objects of $\mathbf{S}_m$ contains the set of values of m. That allows for a nice free possibility for interpreting the values of the many-valued context (G, M, W, I), but in practice the choice of the scale has to be carefully connected to the meaning of the values as assigned by the “expert”, which is responsible for the data in (G, M, W, I). Using the Conceptual Ontology Systems we try to make explicit that the values of a many-valued context usually are “concepts” in a certain ontological hierarchy. If that hierarchy is not yet a concept lattice, it can be embedded into a concept lattice in several meaningful ways, for example by taking the Dedekind-MacNeille completion of the given ordered set.

The formal connection between ontology scaling and plain scaling is given by the following lemma.

Lemma 1 ***Ontology Scaling Lemma***
Let M be a set, and for each $m \in M$ let $\mathbb{O}_m := (G_m, N_m, I_m)$ be a formal context with $\mathfrak{B}_m$ as its concept lattice. Let $((\mathbb{O}_m \mid m \in M), (G, M, W, I))$ be a Conceptual Ontology System with $\mathbb{K}^o := (G, \{(m,n) | m \in M, n \in N_m\}, J^o)$ as its ontology derived context. For each $m \in M$ let $\boldsymbol{S}_m := (\mathfrak{B}_m, N_m, J_m)$ where

$$(A,B)J_m n :\Longleftrightarrow n \in B \ (\text{for } (A,B) \in \mathfrak{B}_m).$$

Then the derived context $\mathbb{K}$ of the scaled many-valued context $((G, M, W, I), (\boldsymbol{S}_m | m \in M))$ satisfies $\mathbb{K} = \mathbb{K}^o$.

Proof The formal contexts $\mathbb{K}$ and $\mathbb{K}^o$ have the same set G of formal objects and the same set of formal attributes. Let J denote the incidence relation of $\mathbb{K}$. Then, by definition of the derived context $gJ(m,n) \Longleftrightarrow \exists (A,B) \in \mathfrak{B}_m$ such that $(A,B)J_m n \Longleftrightarrow \exists (A,B) \in \mathfrak{B}_m n \in B \Longleftrightarrow gJ^o(m,n)$. □

We interpret Lemma 1 as follows. If experts in some field have observed some phenomena and have made a protocol of "instances" using concepts of their well-accepted ontologies, then the ontology scaled context of their Conceptual Ontology System is "correctly reconstructed" by the usual plain scaling of the underlying many-valued context, if one uses the scales $\mathbf{S}_m := (\mathfrak{B}_m, N_m, J_m)$ described in Lemma 1.

In the next section we use Conceptual Ontology Systems for investigating particles and waves.

3 Packets, Time, Location

In this section we give a formal definition of "particles" and "waves" in a given Conceptual Ontology System with three specified ontologies describing *types* of "general objects" (or "packets"), "time granules", and "locations" (or "space granules").

3.1 Ontologies for Packets, Time, and Location

Definition 3. *Let $((\mathbb{O}_m \mid m \in M), (G, M, W, I))$ be a Conceptual Ontology System and $P, T, L \in M$. Then the tripel $((\mathbb{O}_m \mid m \in M), (G, M, W, I), (P,T,L))$ is called a spatio-temporal Conceptual Ontology System. Let $\mathbb{K}^o$ denote its ontology derived context, $\mathfrak{B}^o$ its concept lattice and γ^o its object concept mapping. For each $m \in M$ the context $\mathbb{K}^o_m := (G, \{(m,n) | n \in N_m\}, J^o \cap (G \times \{(m,n) | n \in N_m\}))$ is called the m-part of $\mathbb{K}^o$. Its concept lattice is denoted by $\mathfrak{B}^o_m$, its object concept mapping by γ^o_m.*
The formal concepts of the ontologies $\mathbb{O}_P$, $\mathbb{O}_T$, $\mathbb{O}_L$ are called types of "general objects" or "packets", types of "time granules", and types of "space granules" respectively.
The formal concepts of the parts $\mathfrak{B}^o_P$, $\mathfrak{B}^o_T$, $\mathfrak{B}^o_L$ are called "general objects" or "packets", "time granules", and "space granules" respectively.

Definition 4. ***"location of a packet at a time granule"***
Let $\mathfrak{C} := ((\mathbb{O}_m \mid m \in M), (G, M, W, I), (P,T,L))$ be a spatio-temporal COS. Let

$\boldsymbol{p} := (A_{\boldsymbol{p}}, B_{\boldsymbol{p}})$ be a packet of $\mathfrak{C}$, and $\boldsymbol{t} := (A_{\boldsymbol{t}}, B_{\boldsymbol{t}})$ be a time granule of $\mathfrak{C}$. Then the pair ($\boldsymbol{p}$,$\boldsymbol{t}$) is called an actual packet of $\mathfrak{C}$ and the set

$$L(\boldsymbol{p},\boldsymbol{t}) := \{\gamma_L^o(g) | g \in A_{\boldsymbol{p}} \cap A_{\boldsymbol{t}}\}$$

is called the location of ($\boldsymbol{p}$,$\boldsymbol{t}$) in $\mathfrak{C}$ or the location of the packet $\boldsymbol{p}$ at time granule $\boldsymbol{t}$.

In biology, the habitat of a species can be understood as the location of a packet representing the species as a general object. The locations will be used to define particles and waves in the next section.

3.2 Particles and Waves as Special Packets

The following definition introduces the notions of *particles* and *waves* in spatio-temporal Conceptual Ontology Systems.

Definition 5. ***"particles and waves"***
Let $\mathfrak{C}$:= (($\mathbb{O}_m$ | $m \in M$), (G, M, W, I), (P,T,L)) be a spatio-temporal COS. An actual packet ($\boldsymbol{p}$,$\boldsymbol{t}$) of $\mathfrak{C}$ is called

- *not observed, if $|L(\boldsymbol{p}, \boldsymbol{t})| = 0$,*
- *particle-like (or object-like) if $|L(\boldsymbol{p}, \boldsymbol{t})| = 1$,*
- *wave-like if $|L(\boldsymbol{p}, \boldsymbol{t})| \geq 2$.*

A packet $\boldsymbol{p}$ of $\mathfrak{C}$ is called

- *a particle (or an object) of $\mathfrak{C}$ if $|L(\boldsymbol{p}, \boldsymbol{t})| \leq 1$ for all time granules $\boldsymbol{t} \in \gamma_T(G)$,*
- *a wave of $\mathfrak{C}$ if $|L(\boldsymbol{p}, \boldsymbol{t})| \geq 2$ for all time granules $\boldsymbol{t} \in \gamma_T(G)$.*

The set $\gamma_T(G)$ is interpreted as the set of time granules, which are "mentioned in the protocolled instances".

4 Representation of Classical Particles and Waves

In this section we describe first "classical particles" as particles in a spatio-temporal COS, and then "classical waves" as waves in a spatio-temporal COS.

4.1 Representation of Classical Particles

Let P_c, T_c, X_c be sets, interpreted as sets of "classical" particles, time points, and places, respectively. According to the "Object Axiom" we introduce for each $p \in P_c$ a mapping

$$x_p : T_c \to X_c$$

called the *trajectory of p*; $x_p(t)$ is interpreted as the place of p at time t. The structure (P_c, T_c, X_c, ($x_p | p \in P_c$)) is called a *family of classical particles.*

We shall demonstrate, that each family of classical particles can be represented in a spatio-temporal COS $\mathfrak{C}_1$ in such a way, that the classical particles are represented by particles in $\mathfrak{C}_1$. The simple idea for its formal construction can be seen in Table 4.1.

For the formal construction of a suitable spatio-temporal COS $\mathfrak{C}_1$:= (($\mathbb{O}_m$ | $m \in M$), (G, M, W, I), (P,T,L)) we introduce the set $G := P_c \times T_c$, define P, T, L as mappings from G into P_c, T_c, and X_c respectively, by
P(p,t) := p, T(p,t) := t, L(p,t) := $x_p(t)$. Let M := $\{P, T, L\}$; the ontologies are chosen as the nominal scales

Table 1. A data table for classical particles

instances	P	T	L
(p,t)	p	t	$x_p(t)$

$\mathbb{O}_P := (P_c, P_c, =)$, $\mathbb{O}_T := (T_c, T_c, =)$, $\mathbb{O}_L := (X_c, X_c, =)$.

The formal definition of the complete many-valued context (G,M,W,I) of $\mathfrak{C}_1$ is obvious, since P, T, and L are defined as mappings on G; for having concepts of the ontologies as values we replace the values in table 4.1 by their object concepts in the corresponding ontologies and obtain the sets W := $\gamma_P(P_c) \cup \gamma_T(T_c) \cup \gamma_L(X_c)$ and I := $\{((p,t), \gamma_m(m(p,t))|p \in P_c, t \in T_c, m \in M)\}$. The following Lemma 2 shows, that the classical particles are represented as particles in $\mathfrak{C}_1$.

Lemma 2 ***Classical Particle Representation***
Let (P_c, T_c, X_c, $(x_p|p \in P_c)$) be a family of classical particles and $\mathfrak{C}_1$ its corresponding spatio-temporal COS as defined above. Then for each classical particle $p \in P_c$ the packet $\boldsymbol{p}$:= $\mu^o_P(P,p)$ is a particle of $\mathfrak{C}_1$, where $\mu^o_P(P,p)$ denotes the attribute concept of (P,p) in $\mathbb{K}^o_P$.

Proof It is obvious, that $\mathfrak{C}_1$ is a spatio-temporal COS. Let $p \in P_c$. To prove, that the packet $\mathbf{p}$:= $\mu^o_P(P,p)$ is a particle of $\mathfrak{C}_1$, let $t \in T_c$ and $\mathbf{t} := \mu^o_T(T,t)$. Then L($\mathbf{p}$,$\mathbf{t}$)= $\{\gamma^o_L(g)|g \in A_{\mathbf{p}} \cap A_{\mathbf{t}}\} = \{\gamma^o_L(p,t)\}$. Hence $\mathbf{p}$ is a particle in $\mathfrak{C}_1$. □

4.2 Representation of Classical Waves

Similar to the representation of a family of classical particles we can also represent classical waves. The main idea is very simple: Let's assume, that we wish to describe a usual wave, namely a function f defined on the real plane, which assigns to each point (x,y) and to each time point t an amplitude value z := f((x,y),t). For each time point t this function can be described by the "family" of contour lines of all levels of f. And this is the main similarity between a family of classical particles and a wave as a family of contour lines. That leads to a representation of a wave as a packet, which is a superconcept of all object concepts in a nominal ontology for the levels of f. That is explained in the following Lemma 3, which shows, that according to the Wave Axiom each wave, which is described as a function, can be represented in a spatio-temporal COS $\mathfrak{C}_2$ as a wave of $\mathfrak{C}_2$.

Lemma 3 ***Classical Wave Representation***
Let a "classical wave" be described as a function

$$f : X_c \times T_c \to Z_c,$$

where X_c, T_c, Z_c are sets, interpreted as sets of "classical" places, time points, and values, respectively.
For the formal construction of a suitable spatio-temporal COS $\mathfrak{C}_2$:= (($\mathbb{O}_m$ | $m \in M$), (G, M, W, I), (P,T,L)) we introduce the set $G := X_c \times T_c$, define P, T, L as mappings from G into Z_c, T_c, and X_c respectively, by
P(x,t) := f(x,t), T(x,t) := t, L(x,t) := x. Let $M := \{P, T, L\}$; the ontologies are chosen as the nominal scales $\mathbb{O}_P := (Z_c, Z_c, =)$, $\mathbb{O}_T := (T_c, T_c, =)$, $\mathbb{O}_L := (X_c, X_c, =)$.
$W := \gamma_P(Z_c) \cup \gamma_T(T_c) \cup \gamma_L(X_c)$ and $I := \{((x,t), m, \gamma_m(m(x,t))|(x,t) \in G, m \in M)\}$.
Then $\mathfrak{C}_2$ is a spatio-temporal COS. Let $\boldsymbol{p}$ denote the top element in the concept lattice $\mathfrak{B}^o_P$ of the P-part of the ontology derived context $\mathbb{K}^o$.
The $\boldsymbol{p}$ is a wave in $\mathfrak{C}_2$ if $|X_c| \geq 2$. Otherwise, $\boldsymbol{p}$ is a particle.

Proof It is obvious, that $\mathfrak{C}_2$ is a spatio-temporal COS. Since $\mathbf{p}$ is the top element of $\mathfrak{B}_P^o$ its extent $A_{\mathbf{p}} = $ G. To prove, that the packet $\mathbf{p}$ is a wave of $\mathfrak{C}_2$, let $t \in T_c$ and $\mathbf{t} := \mu_T^o(T,t)$. Hence the extent $A_{\mathbf{t}} := ext(\mathbf{t}) = X_c \times \{t\}$. Then L($\mathbf{p}$,$\mathbf{t}$)= $\{\gamma_L^o(g) | g \in A_{\mathbf{p}} \cap A_{\mathbf{t}}\} = \gamma_L^o(A_{\mathbf{t}})$, since $A_{\mathbf{p}} = G$. Hence L($\mathbf{p}$,$\mathbf{t}$)= $\gamma_L^o(X_c \times \{t\})$. Therefore, $|L(\mathbf{p}, \mathbf{t})| \geq 2 \Longleftrightarrow |X_c| \geq 2$. □

In the next section we present three examples, two for classical particles and one for a packet, which sometimes behaves like a wave, and sometimes behaves like a particle. That will be understood as the usual behavior of packets.

5 Examples: Harmonic Oscillator, Practicing Gymnasts, Hare and Hedgehog

The first example represents a simple harmonic oscillator on the unit circle as a particle, the second one represents a well-known gymnastic demonstration for three gymnasts as a family of particles, where their exercises also behave like particles. The third example represents in the German tale "Der Wettlauf zwischen dem Hasen und dem Igel" (The Race Between the Hare and the Hedgehog) the "species" of hedgehogs as a packet, which is neither a particle nor a wave.

For some of these examples we employ in their graphical representations arrows indicating transitions based on the usual linear order on the time set. For the general introduction of transitions in CTSOTs the reader is referred to [17, 21]. Transitions for spatio-temporal COSs in general will be investigated in future research.

5.1 A Harmonic Oscillator

We construct a simple harmonic oscillator as the following family of classical particles $\mathfrak{H} := (P_c,\ T_c,\ X_c,\ (x_p | p \in P_c))$, where $P_c := \{p\}$ has only one element p, $T_c := [0, 2\pi]$, $X_c := \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$, and for $t \in [0, 2\pi]$ let $\mathbf{x}_p(t) := (\cos(t), \sin(t))$. The corresponding spatio-temporal COS $\mathfrak{C}_1(\mathfrak{H})$ clearly has an infinite ontology derived context. For understanding its concept lattice we restrict the set $T_c := [0, 2\pi]$ to the set $T_{12} := \{\frac{2\pi k}{12} | k \in \{0, 1, ..., 12\}\}$. The concept lattice of the ontology derived context of the restricted system is shown in Figure 5.1, which represents the movement of a particle on the unit circle in the situation space, that is the direct product of the state space (here the unit circle) and the time (here the set T_{12}).

5.2 Three Gymnasts: Roll and Jump

We now construct a spatio-temporal COS, which represents a well-known gymnastic demonstration for three gymnasts, say Tobias, Konstantin, and Florian. They start their demonstration at time point 0 standing on a line, Tobias on place1, Konstantin on place2, and Florian on place3. The main information for them is, that the gymnast on the middle place2 rolls to an outer place, while at the same time the gymnast on that outer place whereto the "inner" gymnast rolls, jumps straddling over the rolling gymnast. Each gymnast, which has reached an outer place, has to turn around for the next jump. Konstantin starts rolling to place1 and Tobias jumps straddling to place2. A whole period of this demonstration is shown in Figure 5.2, where the life track of each gymnast is represented by a sequence of arrows, bold for Tobias, middle for Konstantin, and thin for Florian. The formal objects from 1 to 21 are the "instances" of the underlying spatio-temporal COS. The reader will be convinced, that the three gymnasts will be represented as three particles of the

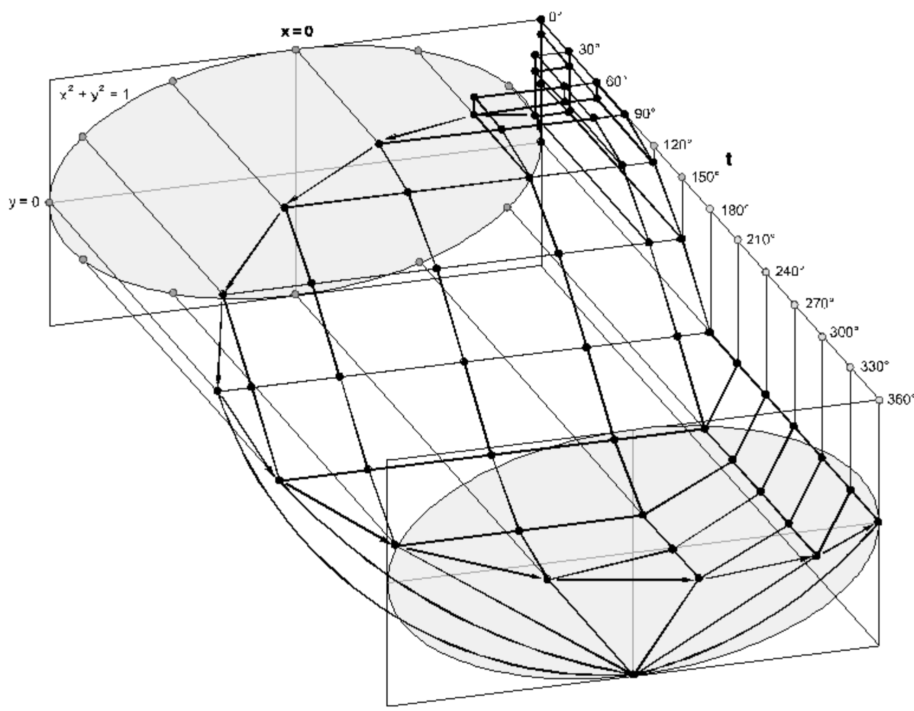

Fig. 1. A harmonic oscillator as a particle moving on the unit circle (drawn by Julia Wiskandt)

underlying spatio-temporal COS. But in this example, there are more formal particles, namely the exercises “roll”, “jump”, and “turn around” yield, as values of the many-valued attribute “exercise”, also particles, if we choose as the “P-part” not the gymnasts, but the exercises. Instead of explaining the details, we just show the corresponding transition diagram for the exercises in Figure 5.2.

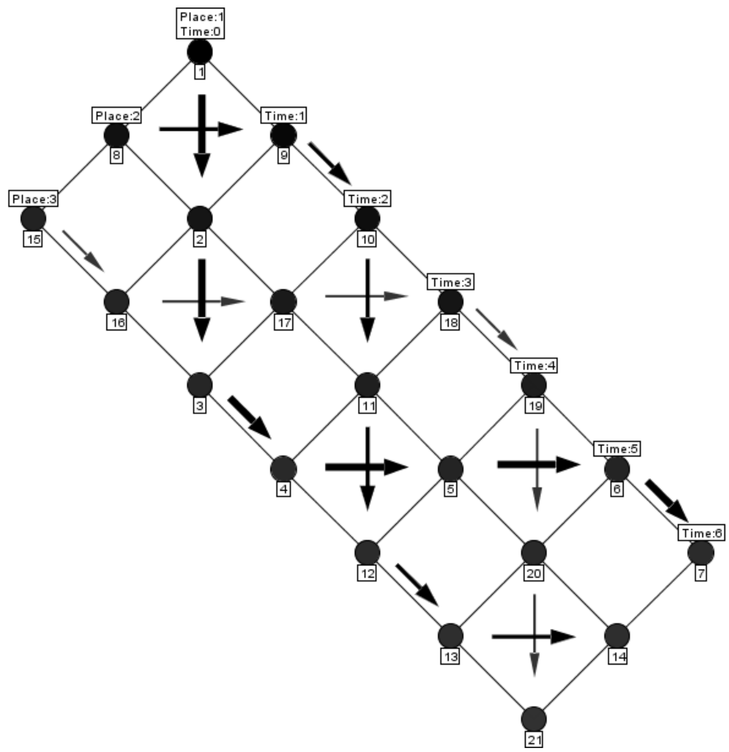

Fig. 2. Tobias (bold arrow), Konstantin (middle), Florian (thin)

For example, instance 9 says, that Konstantin at place1 and time point 1 has turned around; if we now follow the “turn around arrow” to the object concept of the instance 3, we see from Figure 5.2 that someone has turned around at place3 and time point 2, and that is Tobias, since he is the only gymnast, who is at time point 2 on place3.

The observation, that the exercises “behave” sometimes as particles, as in this example, and sometimes as waves, led the author to the present paper. In the next example we will find a typical “wave packet”, which sometimes behaves like a wave, and sometimes like a particle.

5.3 Hare and Hedgehog

The following example has been introduced by the author [20].

> The German tale “Der Wettlauf zwischen dem Hasen und dem Igel” (The Race Between the Hare and the Hedgehog) [1] contains a nice and well-known misinterpretation on the side of the hare, which is based on the fact that the hare does not distinguish between the hedgehog and its wife.

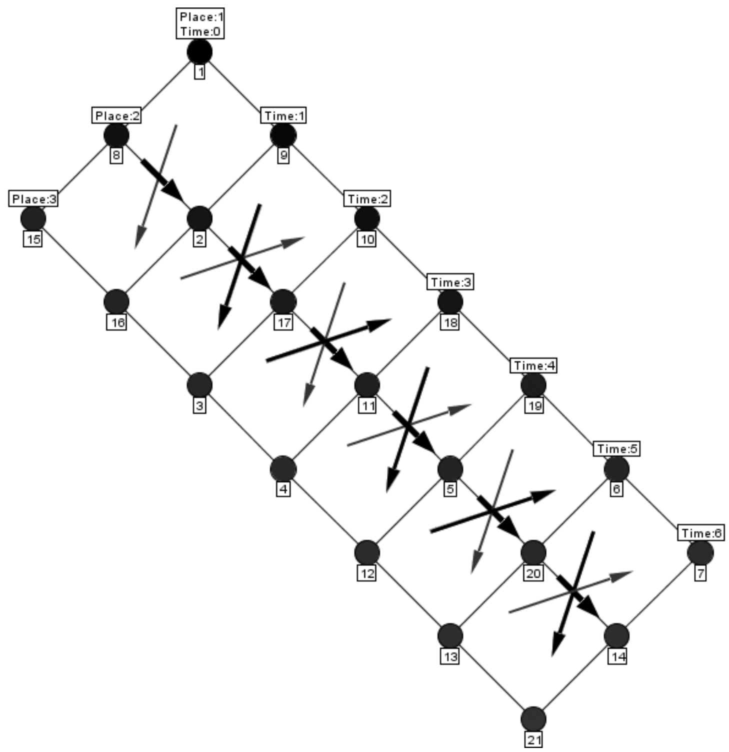

Fig. 3. Exercises as particles: roll(bold arrows), turn around (middle), jump (thin)

A Short Version of the Tale

> On a nice Sunday morning the hedgehog leaves his family for a walk to his field where he meets the hare. They decide to run a race. The hedgehog asks his wife at home to help him in that race by waiting at the lower end of the field. Then the hedgehog starts the race together with the hare at the upper end of the field, but after a few steps the hedgehog stops and returns to the starting point. The hare does not know that and runs down to the lower end of the field where the wife of the hedgehog is waiting. Since the hare does not distinguish her from the hedgehog, he thinks that the hedgehog was quicker. He repeats the race until he breaks down and dies.

Now we demonstrate a simple example of a "wave packet". We make it explicit as a packet in a spatio-temporal COS constructed from the first part of the race between the hare and the hedgehog. For that purpose we introduce ontologies for P := animal, T := time, L := place, taking $\mathbb{O}_P$ as the nominal scale for the set $\{hare, hedgehog\}$, $\mathbb{O}_T$ as the nominal scale for the set $\{0, 1, 2, 3\}$, and the ontology $\mathbb{O}_L$ as given in Table 5.3.

These ontologies together with the many-valued context shown in Table 5.3 form the spatio-temporal COS. For example, in Table 5.3 the value "hedgehog" denotes the attribute concept of the attribute "hedgehog" in $\mathbb{O}_P$. We interpret instance 1 (and instance 2) in Table 5.3 as the statement "An animal of type hedgehog was observed at time 0 in the House." As in many applications multiple observations are protocolled here. That appears also, if we restrict a many-valued context to some part of it, as we did here, since we do not use here the attribute "gender" of the original many-valued context in [20].

Table 2. The ontology $\mathbb{O}_L$ for the places

	House	Field	Upper Field	Lower Field
House	×			
Field		×		
Upper Field		×	×	
Lower Field		×		×

The "hare packet" $\mathbf{p_1} := \mu_P^o(animal, hare)$, that is the attribute concept of (animal,hare) in the P-part of the ontology derived context $\mathbb{K}_P^o$ has at time granule **0**:= $\mu_T^o(time, 0)$ the location L($\mathbf{p_1}$,**0**) = $\emptyset$; that means, that the hare was not mentioned at time granule **0**. But for the other three time granules the hare packet behaves particle-like, hence the hare packet p_1 is a particle in this spatio-temporal COS. Using a time relation as in CTSOTs, we could introduce transitions and life tracks of objects in spatio-temporal COSs. That will be done in a more general setting in a future paper.

The "hedgehog packet" $\mathbf{p_2} := \mu_P^o(animal, hedgehog)$, that is the attribute concept of (animal,hedgehog) in the P-part of the ontology derived context $\mathbb{K}_P^o$ has at time granule **0**:= $\mu_T^o(time, 0)$ the location L($\mathbf{p_2}$,**0**) = $\{\gamma_L^o(1)\}$, hence the actual packet ($\mathbf{p_2}$,**0**) is particle-like, while ($\mathbf{p_2}$,**1**) is wave-like. That can be seen easily in Figure 5.3, where the "oscillating" location of the "hedgehog packet" at the four time granules is indicated by the labels 1,2,4,5,7,8,10,11. That yields a simple conceptual understanding of the phenomena of "generation of particles from waves" and the "decay of particles into waves". Hence, the "hedgehog packet" is a "proper wave packet", that is a packet, which is neither a particle nor a wave.

The example of the "hedgehog packet" can be easily extended to many other examples in daily life. That may help to "de-mystify" the "wave packets" in physics.

Table 3. A many-valued context describing the first part of the race

	P	T	L
instance	animal	time	place
1	hedgehog	0	House
2	hedgehog	0	House
3	hare	1	Field
4	hedgehog	1	Field
5	hedgehog	1	House
6	hare	2	Field
7	hedgehog	2	House
8	hedgehog	2	House
9	hare	3	Upper Field
10	hedgehog	3	Upper Field
11	hedgehog	3	Lower Field

6 Conclusion and Future Research

We have introduced a "symmetric" representation of objects, space, and time in continuously fine or arbitrarily coarse granularity. The intuitive notion of "wave packets" has been conceptually described by the definition of *general objects* or

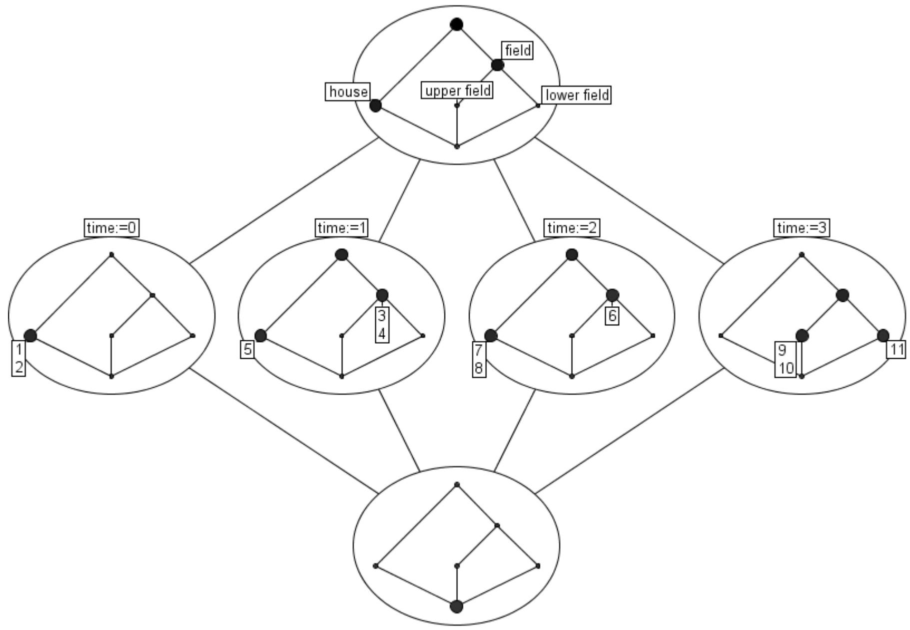

Fig. 4. The "species" hedgehog as a "wave packet"

packets in *spatio-temporal Conceptual Ontology Systems*. That led to a definition of *particles* and *waves*, which covers the "classical" particles and waves in physics. In the examples we have shown, how the developed tools can be used for describing discrete phenomena of movements of general objects.

Future research might focus on the following three topics.

1. The development of a spatio-temporal theory of *general objects* including a generalization of the notions of transitions and life tracks as they have been introduced by the author for the special case of Conceptual Time Systems with Actual Objects and a Time Relation (CTSOT).
2. The investigation of the connection of Conceptual Ontology Systems to Contextual Logic as developed by R. Wille [12].
3. The development of a spatio-temporal logic combining Contextual Logic with the Temporal Logic as developed by D. Gabbay [6] and J. van Benthem [9].

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